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Stationary axisymmetric one-soliton solutions of the Einstein equations

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Abstract. The Belinsky and Zakharov soliton technique is applied to find stationary axisymmetric one-soliton solutions of the Einstein equations in vacuum. In order to generate $(2n+1)$-soliton solutions with physical signature we take the (unphysical) Euclidean metric as the seed solution. The one-soliton solutions are a family of non-asymptotically flat metrics depending on one parameter and can be considered as being the stationary generalisations of a very simple family of static Weyl metrics. They are Petrov type I metrics except for one of its members, which is Petrov type II and can be simply related to the van Stockum class. The Ernst potential of these solutions and the use of prolate spheroidal coordinates suggest new related families of solutions which are asymptotically flat. One of them contains the Zipoy–Voorhees metric with deformation parameter $\delta = \frac{1}{2}$ as a particular case.

1. Introduction

Several generation techniques for finding new solutions of the Einstein equations from known ones when certain symmetries are assumed have been developed in the past few years. For a review of these techniques and their relations see Cosgrove (1980). One such technique, developed by Belinsky and Zakharov (1978), is based on the inverse scattering method (soliton technique) which has been applied to non-linear equations in other fields of physics.

The soliton technique can be applied to the Einstein equations in vacuum if one assumes the existence of two commuting Killing vectors. It allows one to generate the so called n-soliton solutions with an arbitrarily large number of multipole parameters, depending on how large n is, from given 'seed' solutions. Thus, for instance, the Kerr metric is the 2-soliton solution obtained from the Minkowski seed.

The stationary axisymmetric $2n$-soliton solutions from the Minkowski metric have been studied by Belinsky and Zakharov (1980) and Alekseev and Belinsky (1981). Cosgrove (1980) has catalogued all the stationary axisymmetric asymptotically flat metrics that can be deduced from the Weyl and the Cosgrove–Tomimatsu–Sato metrics.

The stationary axisymmetric $(2n+1)$-soliton solutions can only be obtained from a seed with non-physical signature. The reason is that the introduction of one soliton produces a signature change in the metric.

In this paper we study the simplest of such solutions, namely, the stationary axisymmetric one-soliton solutions deduced from the (unphysical) Euclidean metric.

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These solutions can be considered as physical seeds for the general $(2n+1)$-soliton solutions.

The family of metrics obtained depends on two arbitrary parameters (essentially one) and is of Petrov type I except for one of its members. It has two interesting limits which we study in some detail: the 'static' and the 'magnetic' limits. The first one is singular on the axis of symmetry and is related to one of the Kinnersley and Kelly (1974) 'extreme Kerr' solutions. The magnetic limit, on the other hand, is of Petrov type II and is related to the van Stockum (1937) class by a simple complex coordinate transformation.

By studying the Ernst (1968) potentials of the solutions and using prolate spheroidal coordinates one can construct related asymptotically flat solutions. One of them includes the Zipoy–Voorhees solution (Voorhees 1970) with deformation parameter $\delta=\frac{1}{2}$ as a particular case.

The paper is divided as follows.

In § 2 we briefly describe the soliton technique and find the one-soliton solution from a seed metric which is the axisymmetric version of the cosmological Kasner solution and includes the Euclidean metric as a particular case. The solution can be seen as the stationary generalisation of a rather simple family of static (non-flat) Weyl solutions which can be obtained by combining two types of flat solutions. We then give the one-soliton solution from the Euclidean seed and its magnetic limit in appropriate coordinate systems.

In § 3 the Riemann curvature tensor of the solution is obtained in a local orthonormal frame. The normal forms of the Weyl tensor are given for the Petrov type I static limit and for the Petrov type II magnetic limit. We discuss some of the intrinsic features of the metrics.

Finally, in § 4 we use the Ernst potentials to relate our solution to other known solutions. Using the symmetry of these potentials in terms of prolate spheroidal coordinates we find some related asymptotically flat solutions. The most interesting of those includes the Zipoy–Voorhees metric with deformation parameter $\delta=\frac{1}{2}$ which can be interpreted as the external field of a rod. The procedure used also suggests a possible physical interpretation for the one-soliton metric.

2. One-soliton solutions

We will first summarise the main results of the Belinsky and Zakharov (1978, 1980) soliton technique.

For a stationary axisymmetric gravitational field the metric can be written as

$$\mathrm{d}s^2=f(\mathrm{d}\rho^2+\mathrm{d}z^2)+g_{AB}\,\mathrm{d}x^A\,\mathrm{d}x^B \qquad (x^0=t, x^1=\varphi;\ A, B=0, 1) \tag{2.1}$$

where the metric coefficient f and the two-dimensional matrix $\boldsymbol{g}$ depend on ρ and z only. One can always choose coordinates ρ and z such that

$$\det \boldsymbol{g}=-\rho^2; \tag{2.2}$$

in this case the Einstein equations in vacuum take the form

$$\boldsymbol{U}_{,\rho}+\boldsymbol{V}_{,z}=0 \tag{2.3}$$

$$(\ln f)_{,\rho}=-\rho^{-1}+(4\rho)^{-1}\operatorname{Tr}(\boldsymbol{U}^2-\boldsymbol{V}^2) \qquad (\ln f)_{,z}=(2\rho)^{-1}\operatorname{Tr}(\boldsymbol{U}\boldsymbol{V}) \tag{2.4}$$

where

$$U = \rho g_{,\rho} g^{-1} \qquad V = \rho g_{,z} g^{-1}. \tag{2.5}$$

Thus, the matrix g is determined by the non-linear system (2.3) independently of f, whereas the coefficient f is determined by the linear system (2.4) once g is known. The soliton technique focuses its attention on the problem of finding a solution for g. The key point is the association to the system (2.3) of the linear 'eigenvalue' problem

$$\left(\partial_z - \frac{2\lambda^2}{\lambda^2+\rho^2}\partial_\lambda\right)\psi = \frac{\rho V - \lambda U}{\lambda^2+\rho^2}\psi \qquad \left(\partial_\rho + \frac{2\lambda\rho}{\lambda^2+\rho^2}\partial_\lambda\right)\psi = \frac{\rho U + \lambda V}{\lambda^2+\rho^2}\psi \tag{2.6}$$

where λ is a complex spectral parameter and $\psi(\lambda, \rho, z)$ is a two-dimensional matrix that verifies

$$g(\rho, z) = \psi(0, \rho, z). \tag{2.7}$$

Given a particular solution of (2.6), say ψ_0, one generates the so called n-soliton solution ψ by purely algebraic operations if one assumes that ψ is the product of ψ_0 and a two-dimensional matrix with n simple poles in the complex λ plane. Then (2.7) shows that one can find the n-soliton solution g if a particular seed solution g_0 of the Einstein equations is given, provided one can integrate (2.6) to find the ψ_0 corresponding to g_0. In general, such an integration is by no means trivial. However, when g_0 is diagonal it can be easily done. Jantzen (1980), for instance, has integrated the corresponding equations (2.6) for several cosmological solutions.

The explicit results are as follows.

One starts with the 'pole trajectories'

$$\mu_k = W_k - z \pm [(W_k - z)^2 + \rho^2]^{1/2} \qquad (W_k = \text{arbitrary constants}) \tag{2.8}$$

and defines the matrix g' by

$$g' = \left(I - \sum_{k=1}^{n} \frac{R_k}{\mu_k}\right) g_0 \tag{2.9}$$

where $(R_k)_{AB} = n_A^{(k)} m_B^{(k)}$ with

$$m_A^{(k)} = m_C^{0(k)} [\psi_0^{-1}(\mu_k, \rho, z)]_{CA} \qquad (m_C^{0(k)} = \text{arbitrary constants})$$

and the $n_A^{(k)}$ can be obtained from

$$\sum_{l=1}^{n} \Gamma_{kl} n_A^{(l)} = \frac{1}{\mu_k} m_C^{(k)} (g_0)_{CA} \qquad \Gamma_{kl} = \frac{m_C^{(k)} (g_0)_{CA} m_A^{(l)}}{\rho^2 + \mu_k \mu_l}.$$

Then g' is a solution of (2.3) but is not a solution of the Einstein equations since it does not verify (2.2). In fact it can be shown that

$$\det g' = (-1)^n \rho^{2n} \left(\prod_{k=1}^{n} \frac{1}{\mu_k^2}\right) \det g_0; \tag{2.10}$$

however, it is easy to obtain from it a solution

$$g = -\rho(-\det g')^{-1/2} g'. \tag{2.11}$$

Expression (2.10) shows that only for n even will the signature of g_0 and g coincide. For n odd one will need a non-physical seed to obtain a metric with the physical signature.

Turning now to the coefficient f, it is a remarkable fact that, for the n-soliton solution given above, f can be integrated explicitly. In terms of the corresponding seed solution f_0, this coefficient is given by

$$f=Cf_0\rho^{-n^2/2}\left(\prod_{k=1}^{n}\mu_k\right)^{n+1}\left[\prod_{\substack{k,l=1\\k>l}}^{n}(\mu_k-\mu_l)^2\right]^{-1}\det\Gamma_{kl}\qquad(C=\text{arbitrary constant})\tag{2.12}$$

where the square bracket is 1 for $n=1$.

The n-soliton solution can be considered as the one-soliton solution obtained from the $(n-1)$-soliton solution seed. As we increase n, the number of arbitrary parameters of the solution increases. In this sense the simplest solution generated by the Belinsky and Zakharov method is the one-soliton solution obtained from the Euclidean metric.

We take the seed metric

$$f_0=\rho^{s_1^2+s_2^2-1}\qquad g_0=\operatorname{diag}(\rho^{2s_1},\rho^{2s_2})\qquad(s_1+s_2=1)\tag{2.13}$$

which is a solution of (2.3) and (2.4) with the non-physical signature $\det g_0=\rho^2$. It corresponds to the Kasner solution in a cosmological type metric, with z and t as non-ignorable coordinates, and includes the Euclidean metric as the particular case $s_1=0$.

The particular solution ψ_0 from (2.6) is easily found to be

$$\psi_0=\operatorname{diag}[(\rho^2-2z\lambda-\lambda^2)^{s_1},(\rho^2-2z\lambda-\lambda^2)^{s_2}]$$

and using (2.9), (2.11) and (2.12) for $n=1$ we obtain the one-soliton solution

$$\begin{aligned}ds^2=&\frac{C\rho^{2q^2}\cosh(q\psi+D)}{(z^2+\rho^2)^{1/2}}(d\rho^2+dz^2)+\frac{1}{\cosh(q\psi+D)}\\&\times[-\rho^{2s_1}\sinh(s_1\psi+D)\,dt^2-\rho^{2s_2}\sinh(s_2\psi-D)\,d\varphi^2-2\rho\cosh(\tfrac{1}{2}\psi)\,dt\,d\varphi]\end{aligned}\tag{2.14}$$

where C and D are arbitrary constants, $q=s_1-\frac{1}{2}$ and

$$e^{-\psi}=(\rho/\mu)^2\qquad\text{with }\mu=W-z+[(W-z)^2+\rho^2]^{1/2}.\tag{2.15}$$

By a shift in the origin of the z axis one can always eliminate the arbitrary constant W. Only the parameter D is essential. Choosing the negative sign in front of the pole trajectory (2.15) leads to the same metric after a coordinate transformation.

The solution (2.14) corresponds to the Belinsky and Zakharov (1978) one-soliton cosmological solution generated from the Kasner metric. Notice, however, that a simple complex coordinate transformation from that solution would lead to a metric with a non-physical signature.

It is interesting to look at the matrix element g_{tt} since it corresponds to the real part of the Ernst (1968) potential $\mathscr{E}$,

$$\operatorname{Re}\mathscr{E}=\rho^{2s_1}\frac{\sinh(s_1\psi+D)}{\cosh(q\psi+D)}.\tag{2.16}$$

Changing now the coordinates (ρ, z) to (r,θ),

$$\rho=r\sin\theta\qquad z-W=r\cos\theta\qquad(0\leqslant\theta\leqslant\pi),\tag{2.17}$$

the function ψ will depend only on θ through the combination

$$\frac{\rho}{\mu}=\left(\frac{1+\cos\theta}{1-\cos\theta}\right)^{1/2}. \tag{2.18}$$

The Euclidean case ($s_1=0$) is particularly simple since Re $\mathscr{E}$ will depend on $\cos\theta$ only. In § 4 we will relate these to the Kinnersley and Kelly (1974) 'extreme Kerr' solutions.

The 'static' limit ($D\to\infty$) is also interesting since (2.14) reduces to the trivial family of Weyl (non-flat) solutions

$$\mathscr{E}=\mu\rho^{2s_1-1} \tag{2.19}$$

which are obtained by combining the flat solutions $\mathscr{E}=\mu$ and $\mathscr{E}=\rho$ (Kramer *et al* 1980). Note that $s_1=\frac{1}{2}$ corresponds to flat space. The one-soliton solution (2.14) can thus be seen as the stationary generalisations of the particular static Weyl family (2.19).

From now on our attention will concentrate on the solution corresponding to the Euclidean seed ($s_1=0$). Using (r,θ) coordinates such a solution can be written as

$$\mathrm{d}s^2=\frac{B(\gamma+\cos\theta)}{r^{1/2}\sin^{1/2}\theta}(\mathrm{d}r^2+r^2\,\mathrm{d}\theta^2)+\frac{1}{\gamma+\cos\theta}$$
$$\times[-\sin\theta\,\mathrm{d}t^2+(1+2\gamma\cos\theta+\cos^2\theta)r^2\sin\theta\,\mathrm{d}\varphi^2-2\varepsilon r\sin\theta\,\mathrm{d}t\,\mathrm{d}\varphi] \tag{2.20}$$

where

$$\gamma=\coth(D/2)\qquad \varepsilon=(\gamma^2-1)^{1/2}\qquad B=\text{arbitrary constant}. \tag{2.21}$$

The static limit of that family of solutions corresponds now to $\gamma=1$. Another interesting limit, which we will call the 'magnetic' limit, is $\gamma\to\infty$ keeping $B\gamma=C$ finite. In this magnetic limit the metric adopts the form

$$\mathrm{d}s^2=C\rho^{-1/2}(\mathrm{d}\rho^2+\mathrm{d}z^2)-2\rho\,\mathrm{d}t\,\mathrm{d}\varphi+2(z-W)\rho\,\mathrm{d}\varphi^2 \tag{2.22}$$

which corresponds to a member of the van Stockum (1937) class. In fact, the complex coordinate change $t\to\mathrm{i}\varphi$, $\varphi\to\mathrm{i}t$ puts (2.22) in the usual van Stockum form (Kinnersley 1975).

3. Classification of the solutions

In this section we will classify the one-soliton solutions generated from Euclidean space, and some of their intrinsic properties will be analysed. The notation throughout this and the next section will be that found in Kramer *et al* (1980).

First we evaluate the Riemann curvature tensor in a local orthonormal frame. The metric is given by

$$\mathrm{d}s^2=\eta_{ab}\boldsymbol{\omega}^a\boldsymbol{\omega}^b\qquad (a,b=0,1,2,3) \tag{3.1}$$

where $\eta_{ab}=\mathrm{diag}(-1,1,1,1)$.

The connection 1-forms $\boldsymbol{\Gamma}^a{}_b$ are given by

$$\mathrm{d}\boldsymbol{\omega}^a=-\boldsymbol{\Gamma}^a{}_b\wedge\boldsymbol{\omega}^b$$

and the curvature 2-forms $\Theta^a{}_b$ by

$$\Theta^a{}_b = \mathrm{d}\Gamma^a{}_b + \Gamma^a{}_c \wedge \Gamma^c{}_b = \tfrac{1}{2} R^a{}_{bcd}\, \omega^c \wedge \omega^d \tag{3.2}$$

whence the components of the Riemann curvature tensor $R^a{}_{bcd}$, in the basis (3.1), can be read.

In an orthonormal basis and in vacuum the components $R_{(ab)(cd)}$ of the Riemann tensor, which is identical to the Weyl tensor, can be written in a 6×6 symmetric matrix $\boldsymbol{R}$ composed of symmetric 'electric', $\boldsymbol{E}$, and symmetric 'magnetic', $\boldsymbol{B}$, 3×3 matrices as

$$\boldsymbol{R} = \begin{pmatrix} \boldsymbol{E} & \boldsymbol{B} \\ \boldsymbol{B} & -\boldsymbol{E} \end{pmatrix} \tag{3.3}$$

with

$$\mathrm{Tr}(\boldsymbol{E}) = \mathrm{Tr}(\boldsymbol{B}) = 0. \tag{3.4}$$

For the components of $\boldsymbol{R}$ we will use the convention: $(01)=1$, $(02)=2$, $(03)=3$, $(23)=4$, $(31)=5$ and $(12)=6$.

The orthonormal basis that we take for the metric (2.20) is

$$\begin{aligned} \omega^0 &= \sin^{1/2}\theta(\gamma+\cos\theta)^{1/2}(1+2\gamma\cos\theta+\cos^2\theta)^{-1/2}\,\mathrm{d}t \\ \omega^1 &= \sin^{1/2}\theta(\gamma+\cos\theta)^{-1/2}(1+2\gamma\cos\theta+\cos^2\theta)^{1/2} r\,\mathrm{d}\varphi \\ &\qquad -\varepsilon\sin^{1/2}\theta(\gamma+\cos\theta)^{-1/2}(1+2\gamma\cos\theta+\cos^2\theta)^{-1/2}\,\mathrm{d}t \\ \omega^2 &= B^{1/2} r^{-1/4}\sin^{-1/4}\theta(\gamma+\cos\theta)^{1/2}\,\mathrm{d}r \qquad \omega^3 = B^{1/2} r^{3/4}\sin^{-1/4}\theta(\gamma+\cos\theta)^{1/2}\,\mathrm{d}\theta. \end{aligned} \tag{3.5}$$

For a metric of type (2.1) the components of the Riemann tensor verify

$$E_{12} = E_{13} = B_{12} = B_{13} = 0. \tag{3.6}$$

The remaining components can be found from (3.2), after a straightforward calculation, to be

$$\begin{aligned} E_{11}/a &= \sin^{-2}\theta(\gamma+\cos\theta)^{-2}(\gamma^2-2+3\cos^2\theta+2\gamma\cos^3\theta) \\ E_{33}/a &= 1-\tfrac{1}{2}\sin^{-2}\theta\cos^2\theta-\tfrac{3}{2}\cos\theta(\gamma+\cos\theta)^{-1}+(\gamma+\cos\theta)^{-2}(\gamma^2-2+\cos^2\theta) \\ &\qquad +(2+\gamma\cos\theta-\cos^2\theta)(1+2\gamma\cos\theta+\cos^2\theta)^{-1} \\ E_{23}/a &= -\tfrac{3}{2}\sin^{-1}\theta(\gamma+\cos\theta)^{-1}(\gamma+\varepsilon+\cos\theta)^{-1}\{\gamma-\varepsilon+[\gamma(\gamma+\varepsilon)+1]\cos\theta \\ &\qquad +(\gamma+2\varepsilon)\cos^2\theta\} \\ B_{11}/ab &= -2\sin\theta(\gamma+\cos\theta)^{-1} \\ B_{33}/ab &= -\tfrac{1}{2}\sin^{-1}\theta(\gamma+\cos\theta)^{-1}(4+3\gamma\cos\theta-\cos^2\theta) \\ &\qquad +3\sin\theta(\gamma+\cos\theta)(1+2\gamma\cos\theta+\cos^2\theta)^{-1} \\ B_{23}/ab &= -\tfrac{3}{2}(3+4\gamma\cos\theta+\cos^2\theta)(1+2\gamma\cos\theta+\cos^2\theta)^{-1} \end{aligned} \tag{3.7}$$

where

$$a = (4B)^{-1} r^{-3/2}\sin^{-1/2}\theta(\gamma+\cos\theta)^{-1} \qquad b = \varepsilon(\gamma+\cos\theta)^{-1}.$$

By projecting the Riemann tensor on a complex null tetrad, easily constructed from the orthonormal frame (3.5), and using the d'Inverno and Russell–Clark algorithm as given in Kramer *et al* (1980), it is not difficult to see that our metric for a finite γ is of Petrov type I and therefore admits in general four principal null directions.

From the scalar invariants that one can construct with the Riemann tensor it is easy to see that the metric has intrinsic singularities at the values of θ corresponding to the zeros of $\sin\theta$, $\gamma+\cos\theta$ (only for $\gamma=1$) and $\gamma-\varepsilon+\cos\theta$. These include the axis of symmetry and a 'cone' of angle $\pi/2\leq\theta\leq\pi$ which coincides with the axis for the static case ($\gamma=1$) and with the $z=W$ plane for the magnetic limit ($\gamma\to\infty$).

Our field is, therefore, not asymptotically flat. On the non-singular directions of θ it becomes flat at $r\to\infty$ as $r^{-3/2}$; this is a milder flattening than, for instance, the Schwarzschild solution which goes like r^{-3}.

The Riemann tensor becomes particularly simple in the static and magnetic limits. In both cases it is not difficult to find the normal forms of such a tensor.

For the static limit ($\gamma=1$) the magnetic part of the curvature tensor vanishes ($\boldsymbol{B}=0$) and the remaining electric part is

$$\boldsymbol{E}=(8B)^{-1}r^{-3/2}\sin^{-1}\theta(1+\cos\theta)^{-1}\begin{pmatrix}-2+4\cos\theta & 0 & 0\\ 0 & -2+\cos\theta & -3\sin\theta\\ 0 & -3\sin\theta & 4-5\cos\theta\end{pmatrix} \tag{3.8}$$

which can be easily diagonalised to give it in the normal form. The eigenvalues are proportional to $-2+4\cos\theta$, $1-2\cos\theta\pm3\sqrt{3}(1-\cos\theta)^{1/2}$ and are all different, except in two directions of θ, confirming the character of a Petrov type I metric. The intrinsic singularity reduces in this case to the axis of symmetry.

For the magnetic limit ($\gamma\to\infty$) the metric is of Petrov type II as one can expect from its relation to the van Stockum metric (Hoffman 1969). After a rotation of axis one can write $\boldsymbol{E}+\mathrm{i}\boldsymbol{B}$ in the normal form

$$\boldsymbol{E}'+\mathrm{i}\boldsymbol{B}'=3(8C)^{-1}\rho^{-1/2}(z-W)^{-1}\begin{pmatrix}\lambda & 0 & 0\\ 0 & -\frac{1}{2}\lambda+1 & \mathrm{i}\\ 0 & \mathrm{i} & -\frac{1}{2}\lambda-1\end{pmatrix}\qquad \lambda=\tfrac{2}{3}(z-W)\rho^{-1}. \tag{3.9}$$

Besides the singularity on the axis of symmetry, the plane $z=W$ is also singular.

Because of their singularities, these metrics cannot represent the external fields of bounded objects. But their interpretation as limiting metrics valid in the vicinity of some singular surface of an asymptotically flat metric cannot be disregarded, in principle, as we will briefly discuss in the next section.

4. Related asymptotically flat solutions

The study of stationary axisymmetric fields in terms of the Ernst (1968) formulation has been most fruitful in the search for new solutions and for their physical interpretation. Here we will consider the Ernst formulation in order to relate the one-soliton solutions (2.20) to other known solutions. We will see also that the simplicity of the relevant Ernst potentials suggests new related families of asymptotically flat solutions.

For such a formulation, the metric (2.1) is better written in the form

$$\mathrm{d}s^2=F^{-1}[\mathrm{e}^{2K}(\mathrm{d}\rho^2+\mathrm{d}z^2)+\rho^2\,\mathrm{d}\varphi^2]-F(\mathrm{d}t+A\,\mathrm{d}\varphi)^2. \tag{4.1}$$

The Ernst potential is a complex function

$$\mathscr{E}=F+\mathrm{i}\omega \tag{4.2}$$

in terms of which the Einstein equations in vacuum become (see Kramer *et al* 1980)

$$(\mathscr{E}+\mathscr{E}^*)\nabla^2\mathscr{E}=2\nabla\mathscr{E}\cdot\nabla\mathscr{E} \tag{4.3}$$

$$(\mathscr{E}+\mathscr{E}^*)^2K_{,\zeta}=\sqrt{2}\rho\mathscr{E}_{,\zeta}\mathscr{E}^*_{,\zeta} \qquad (\mathscr{E}+\mathscr{E}^*)^2A_{,\zeta}=2\rho(\mathscr{E}-\mathscr{E}^*)_{,\zeta} \tag{4.4}$$

where $(\ )^*$ indicates complex conjugation, $\nabla=(\partial_\rho,\partial_z)$, $\nabla^2=\partial_\rho^2+\rho^{-1}\partial_\rho+\partial_z^2$ and $\sqrt{2}\,\partial_\zeta=\partial_\rho-\mathrm{i}\partial_z$.

Now, comparing (4.1) with (2.20) and using (4.4) to find ω, we get the Ernst potential for the one-soliton solution, which, in terms of θ, is

$$\mathscr{E}=\frac{1}{\gamma+\cos\theta}(\sin\theta+\mathrm{i}\varepsilon\gamma^{-1}\cos\theta). \tag{4.5}$$

The fact that this potential depends only on the variable $\cos\theta$ can be exploited further. But for that, it is better to use another Ernst potential ξ defined from $\mathscr{E}$ by

$$\xi=\frac{1-\mathscr{E}}{1+\mathscr{E}}. \tag{4.6}$$

It is worth noting that the real and imaginary parts of this potential have been interpreted by Kinnersley and Kelly (1974) as representing the relativistic version of the Newtonian and magnetic potentials, respectively. In that sense $\varepsilon\gamma^{-1}$ represents a 'magnetic charge': it takes the value $\varepsilon\gamma^{-1}=0$ in the static limit and its maximum value, $\varepsilon\gamma^{-1}=1$, in the magnetic limit.

One of the most important features of ξ is that, by using prolate spheroidal coordinates (x, y), which are related to the Weyl coordinates (ρ, z) by

$$\rho=\sigma(x^2-1)^{1/2}(1-y^2)^{1/2} \qquad z=\sigma xy \qquad \sigma=\text{constant}$$
$$2\sigma x=r_++r_- \qquad 2\sigma y=r_+-r_- \qquad r_\pm^2=\rho^2+(z\pm\sigma)^2, \tag{4.7}$$

the Ernst equation (4.3) becomes

$$(\xi\xi^*-1)\{[(x^2-1)\xi_{,x}]_{,x}+[(1-y^2)\xi_{,y}]_{,y}\}=2\xi^*[(x^2-1)\xi_{,x}^2+(1-y^2)\xi_{,y}^2]. \tag{4.8}$$

The symmetry of this equation in the coordinates x and y has been exploited by Tomimatsu and Sato (1972) to find new solutions.

Now, since the (x, y) coordinates can be written in terms of the Boyer–Lindquist coordinates (R, Θ) (which coincide with our coordinates (2.17) (r, θ) when R is large) as

$$\sigma x=R-m \qquad y=\cos\Theta, \tag{4.9}$$

equation (4.5) suggests the Ernst potential in prolate spheroidal coordinates

$$\mathscr{E}=\frac{(1-y^2)^{1/2}+\mathrm{i}\varepsilon\gamma^{-1}y}{\gamma+y} \tag{4.10}$$

which provides a solution $\xi(y)$ for (4.8).

From any given solution $\xi(x, y)$ of (4.8) one can construct a new solution by commuting x and y. Therefore, from (4.10) we can construct the new solution

$$\mathscr{E}=\mathrm{i}\frac{(x^2-1)^{1/2}+\varepsilon\gamma^{-1}x}{\gamma+x} \tag{4.11}$$

and also, as it is easy to check, the static solution

$$\mathscr{E} = \frac{(x^2-1)^{1/2} - \varepsilon\gamma^{-1}x}{\gamma + x}. \tag{4.12}$$

Both solutions are asymptotically flat. The last one (4.12) contains the Zipoy–Voorhees metric

$$\mathscr{E} = \left(\frac{x-1}{x+1}\right)^{\delta} \tag{4.13}$$

with deformation parameter $\delta = \frac{1}{2}$ when $\gamma = 1$, and a physical interpretation for it can be given.

In fact, Voorhees (1970) gave a plausible physical interpretation for the family (4.13) by comparing the family with the member $\delta = 1$ which is the Schwarzschild metric. For the Schwarzschild solution the coordinates (R, Θ) with $\sigma = m$ can be seen as spherical coordinates, and (4.13) gives the field of a point particle of mass m. In general the coordinates 'adapted to the source' have $\sigma = m/\delta$, and by expanding $\mathscr{E}$ in terms of (R, Θ) for large R, and comparing with the (spherical) Schwarzschild coordinates $(\sigma = m)$, the metrics (4.13) can be interpreted as the external fields of rods (if $\delta < 1$) of mass m. For $\delta = \frac{1}{2}$ the rod has length $4m$.

For large x and finite γ (4.12) can be expanded as

$$\mathscr{E} = 1 - \varepsilon\gamma^{-1} - (1 - \varepsilon\gamma^{-1})\gamma\frac{1}{x} + [(1 - \varepsilon\gamma^{-1})\gamma^2 - \tfrac{1}{2}]\frac{1}{x^2} + \dots.$$

This asymptotic form can be reduced to the usual

$$\mathscr{E} = 1 - \frac{2m}{r} + (\text{polynomial in } \cos\theta)\frac{1}{r^2} + \dots$$

by means of an Ehlers transformation (Cosgrove 1980) which will involve the parameter γ. Therefore, the solution (4.12) can be seen, at least asymptotically, as an Ehlers transformation, depending on γ, of the Zipoy–Voorhees metric with deformation parameter $\delta = \frac{1}{2}$.

The relation between the asymptotically flat solutions and the non-asymptotically flat solution (4.10) is similar to that between the Zipoy–Voorhees metrics and the Kinnersley and Kelly (1974) 'extreme Kerr' solutions representing a region of the Tomimatsu and Sato (1972) metric near to its ergosphere. This seems to suggest that a similar interpretation might be found for the one-soliton solutions as describing some limited region of the external field of rods, at least in the $\gamma = 1$ limit.

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